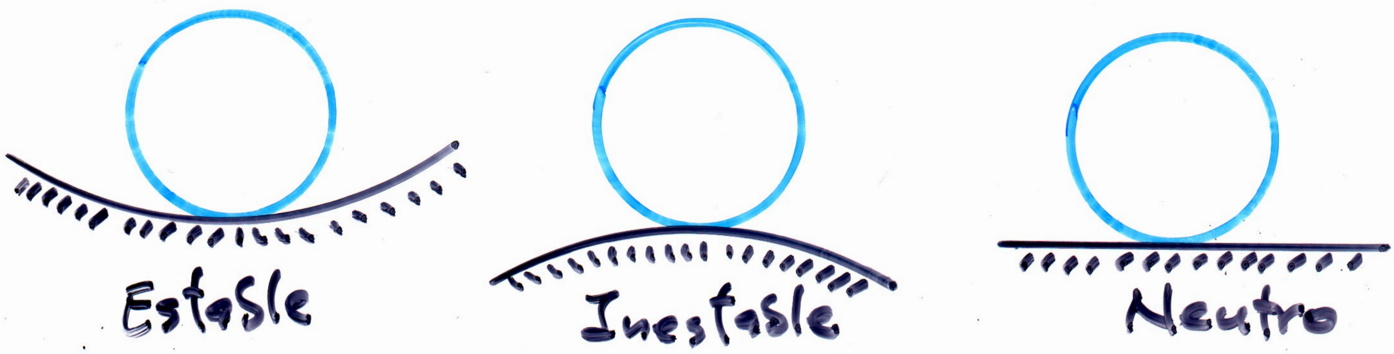
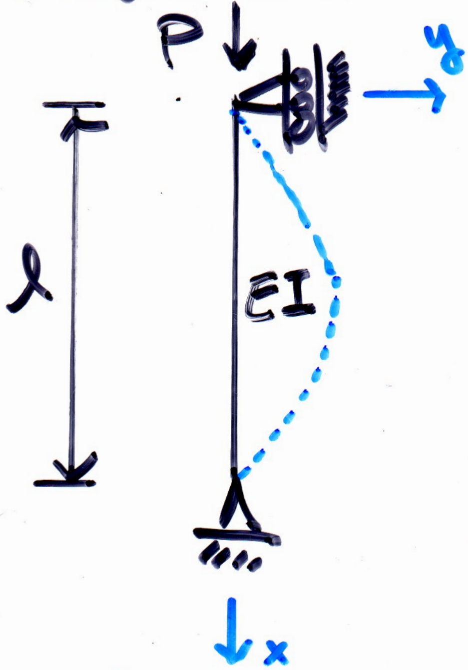
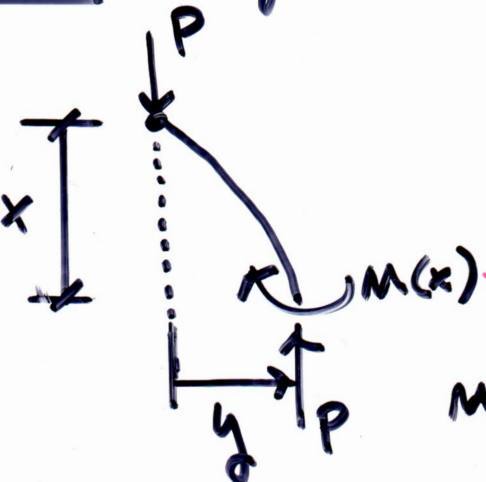
Estabilidad Elástica

Definición: Si un miembro estructural está en equilibrio con sus cargas, y si al imponer una deformación pequeña el miembro tiende a volver a su posición original se dice que el equilibrio es estable. Si en cambio, la deformación aumenta, se dice que el equilibrio es inestable; y si se mantiene el equilibrio en la nueva configuración deformada, se dice que el equilibrio es neutro.



Carga crítica (P_{cr}): Carga de Euler.





momento restaurador

Si:

$M(x) > Py \Rightarrow$ Estable

$M(x) < Py \Rightarrow$ Inestable

$M(x) = Py \Rightarrow$ Neutro

CASO 1

La transición entre el equilibrio estable y el equilibrio inestable ocurre a un valor de carga axial conocido como carga crítica ó carga de pandeo (P_{cr}).

P_{cr} = carga para la cual el equilibrio es neutro

$$M(x) = Py \rightsquigarrow M(x) = -EI\frac{d^2y}{dx^2} \rightarrow \text{curvatura "K"}$$

$$\Rightarrow EI\frac{d^2y}{dx^2} + Py = 0$$

↳ Ecuación Diferencial homogénea, lineal, de 2do orden con coeficientes constantes.

$$\Rightarrow \frac{d^2y}{dx^2} + \underbrace{\frac{P}{EI}}_{K^2}y = 0 \Rightarrow \frac{d^2y}{dx^2} + K^2y = 0$$

Solución: $y = C_1 \sin Kx + C_2 \cos Kx$

Condiciones de Borde: $x = 0$, $y = 0$ ①

$x = l$, $y = 0$ ②

① $0 = C_2$

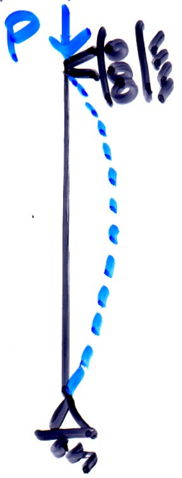
② $0 = C_1 \sin Kl \rightsquigarrow C_1 \neq 0$

$$\Rightarrow \sin Kl = 0$$

$$Kl = \pi,\ 2\pi,\ 3\pi,\ \dots,\ n\pi \rightsquigarrow n = 1, 2, \dots$$

$$\sqrt{\frac{P}{EI}}\,l = n\pi \rightsquigarrow \boxed{P_{cr} = \frac{(n\pi)^2 EI}{l^2}}$$

$n=1$



$n=2$

$n=3$

$$P_{cr} = \frac{\pi^2 EI}{l^2} \qquad P_{cr} = \frac{4\pi^2 EI}{l^2} \qquad P_{cr} = \frac{9\pi^2 EI}{l^2}$$

$P < P_{cr} \rightsquigarrow$ La columna es estable

$P > P_{cr} \rightsquigarrow$ " " " inestable (Pandeo)

<u>Esfuerzo Crítico</u>

r^2

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{(n\pi)^2 EI}{Al^2} = \frac{\pi^2 E}{\left(\frac{l}{nr}\right)^2}$$

minúscula

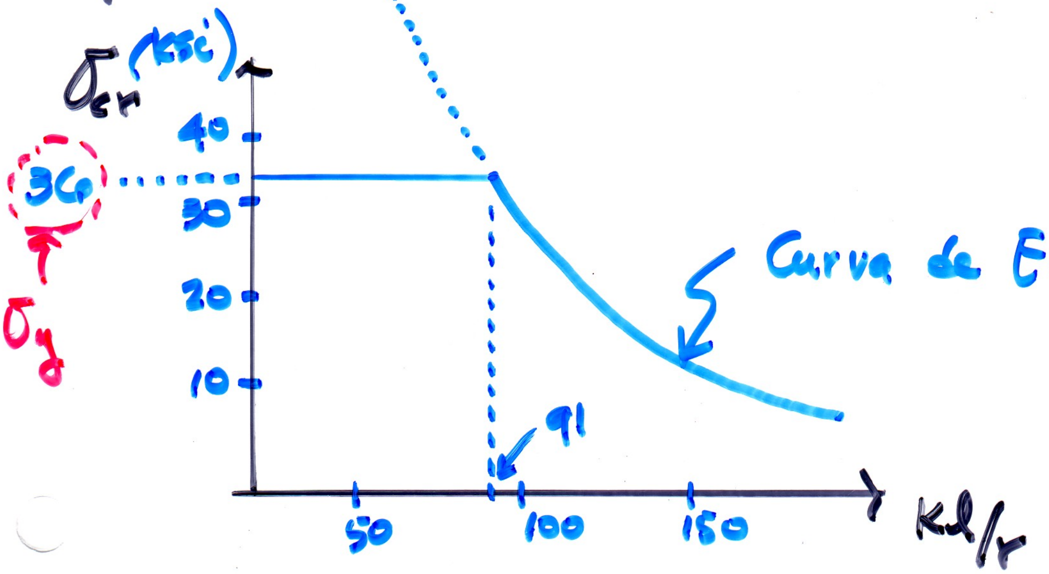
↳ $\frac{1}{n} = k =$ factor de longitud efectiva

$$k \neq K = \sqrt{\frac{P}{EI}} \quad \Rightarrow \quad \boxed{\sigma_{cr} = \frac{\pi^2 E}{(Kl/r)^2}}$$

mayúscula

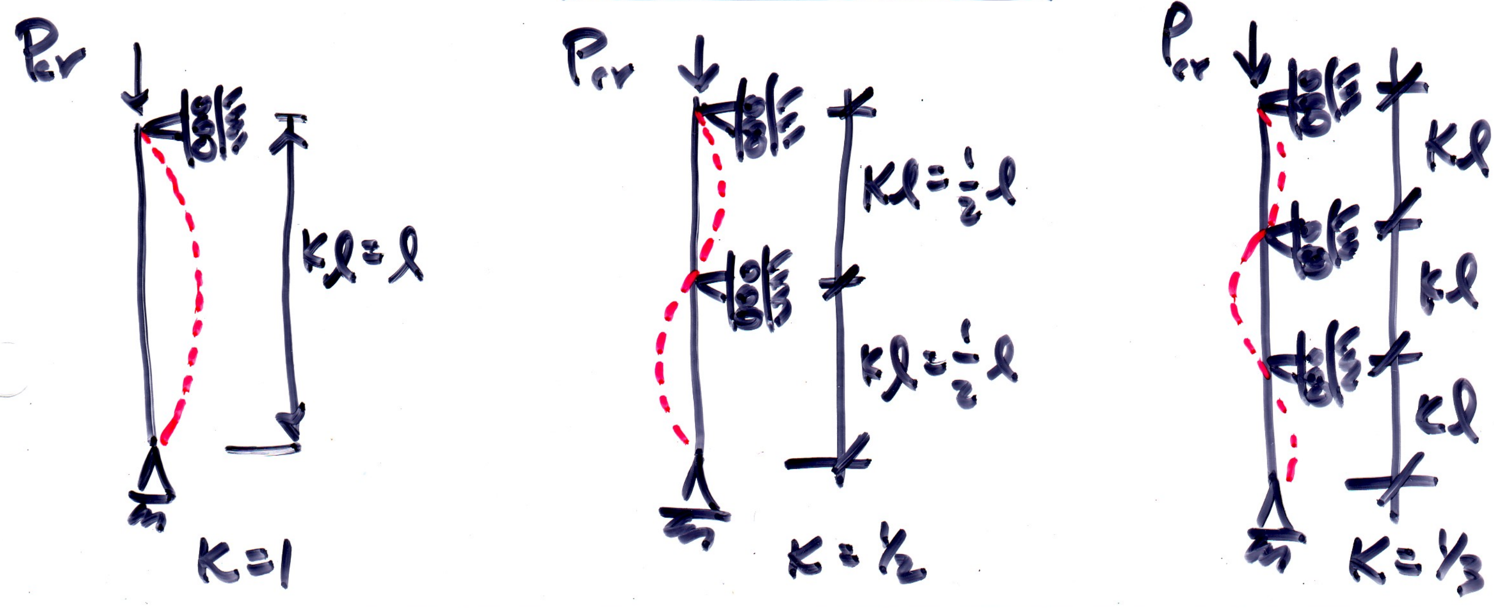
longitud efectiva

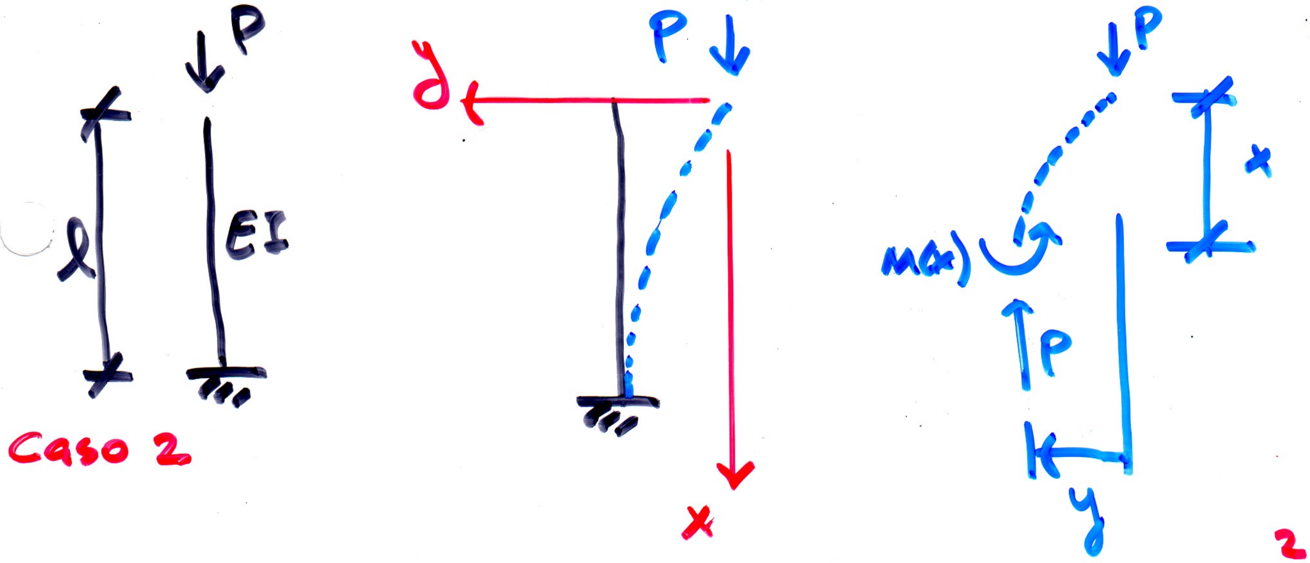
$$\frac{Kl}{r} = \text{razón de esbeltez}$$



Curva de Euler para el Acero Estructural ($\sigma_y = 36^{ksi}$, $E = 29{,}000^{ksi}$)

$$\Rightarrow P_{cr} = \sigma_{cr} \cdot A \sim \boxed{P_{cr} = \frac{\pi^2 AE}{(Kl/r)^2}}$$





$$M(x) = Py$$

$$-EI\frac{d^2y}{dx^2} = Py \Rightarrow \frac{d^2y}{dx^2} + K^2 y = 0 \qquad P/EI = K^2$$

Solución: $y = C_1 \sin Kx + C_2 \cos Kx$

Condiciones de Borde: $x=0,\ y=0$ (1)

$x = l,\ \dfrac{dy}{dx} = 0$ (2)

(1) $0 = C_2$

(2) $\dfrac{dy}{dx} = K C_1 \cos Kx \Rightarrow 0 = K C_1 \cos Kl$ $\quad \neq 0$

$$\Rightarrow \cos Kl = 0 \Rightarrow Kl = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots, \frac{1}{2}(2n-1)\pi$$

↳ $n = 1, 2, 3, \dots$

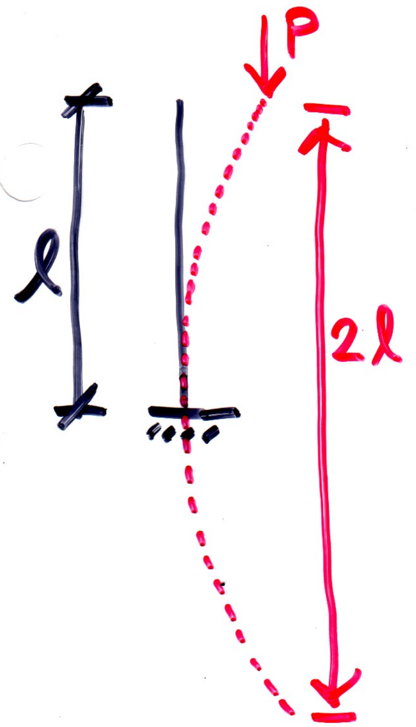
$$\Rightarrow \sqrt{\frac{P}{EI}}\, l = \frac{1}{2}(2n-1)\pi \rightsquigarrow \boxed{P_{cr} = \frac{\left[\frac{\pi}{2}(2n-1)\right]^2 EI}{l^2}}$$

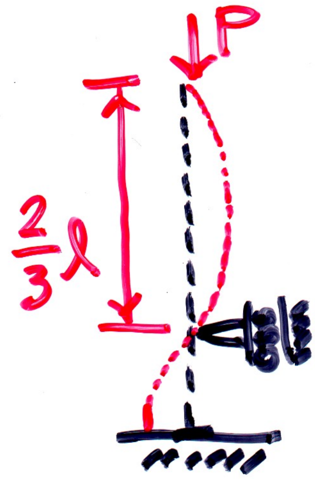
$I = Ar^2$

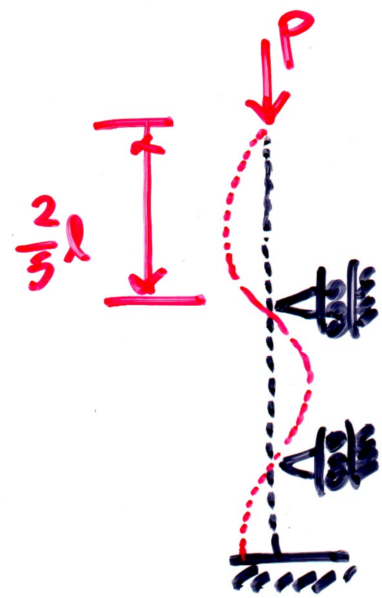
$$P_{cr} = \frac{\pi^2 AE}{\left(\frac{1}{n-\frac{1}{2}} \cdot \frac{l}{r}\right)^2} \sim K = \frac{1}{n-\frac{1}{2}} \Rightarrow \text{factor de long. efectiva}$$

$\Rightarrow Kl$ = longitud efectiva.

n	K	Kl
1	2	$2l$
2	$2/3$	$\frac{2}{3}l$
3	$2/5$	$\frac{2}{5}l$





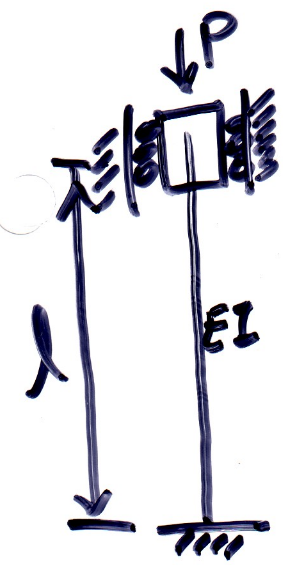


$n = 1$, $Kl = 2l$

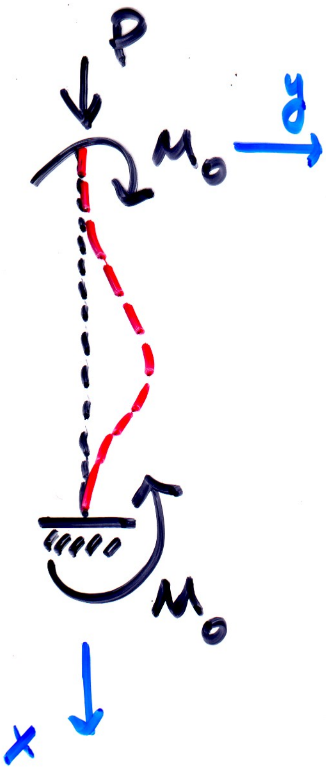
$n = 2$
$Kl = \frac{2}{3}l$

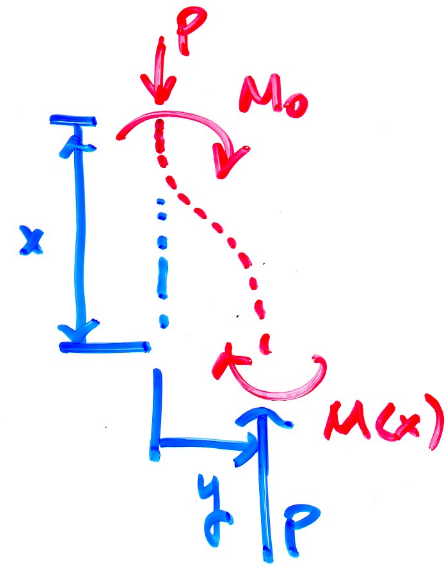
$n = 3$
$Kl = \frac{2}{5}l$

$$P_{cr} = \frac{\pi^2 EI}{(2l)^2} \qquad P_{cr} = \frac{\pi^2 EI}{\left(\frac{2}{3}l\right)^2} \qquad P_{cr} = \frac{\pi^2 EI}{\left(\frac{2}{5}l\right)^2}$$



Caso 3





$$M(x) + M_0 = Py$$

$$EI\frac{d^2y}{dx^2} + Py = M_0$$

$$\frac{d^2y}{dx^2} + K^2 y = \frac{M_0}{EI}$$

Solución Homogénea: $y_h = C_1 \sin Kx + C_2 \cos Kx$

Solución Particular: $y_p = C_3 \rightsquigarrow K^2 C_3 = \frac{M_0}{EI}$

$$\Rightarrow \frac{P}{EI} C_3 = \frac{M_0}{EI} \rightsquigarrow C_3 = \frac{M_0}{P}$$

Solución General: $y = y_h + y_p$

$$y = C_1 \sin Kx + C_2 \cos Kx + \frac{M_0}{P}$$

Condiciones de Borde : 1) $x = 0$, $y = 0$

2) $x = 0$, $\frac{dy}{dx} = 0$

3) $x = \ell$, $y = 0$

$\Rightarrow$ 1) $0 = C_2 + M_0/P \rightsquigarrow C_2 = -M_0/P$

2) $0 = K C_1 \Rightarrow C_1 = 0$

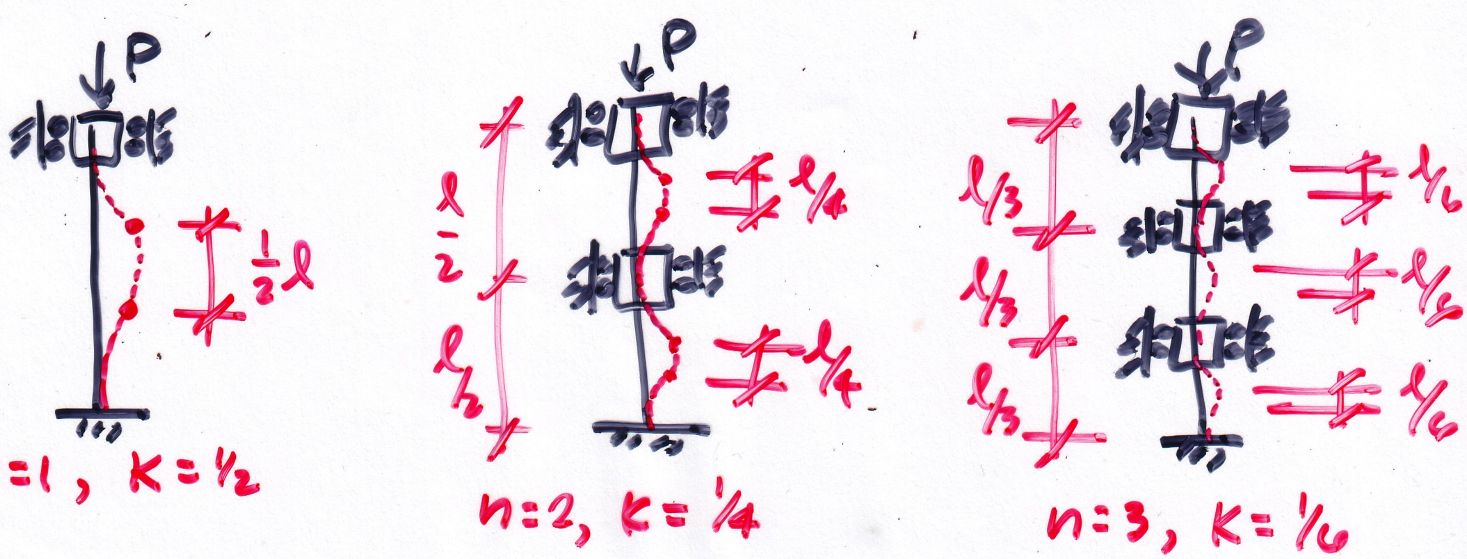
3) $0 = C_2 \cos K\ell + \frac{M_0}{P} \Rightarrow \frac{M_0}{P}(1 - \cos K\ell) = 0$

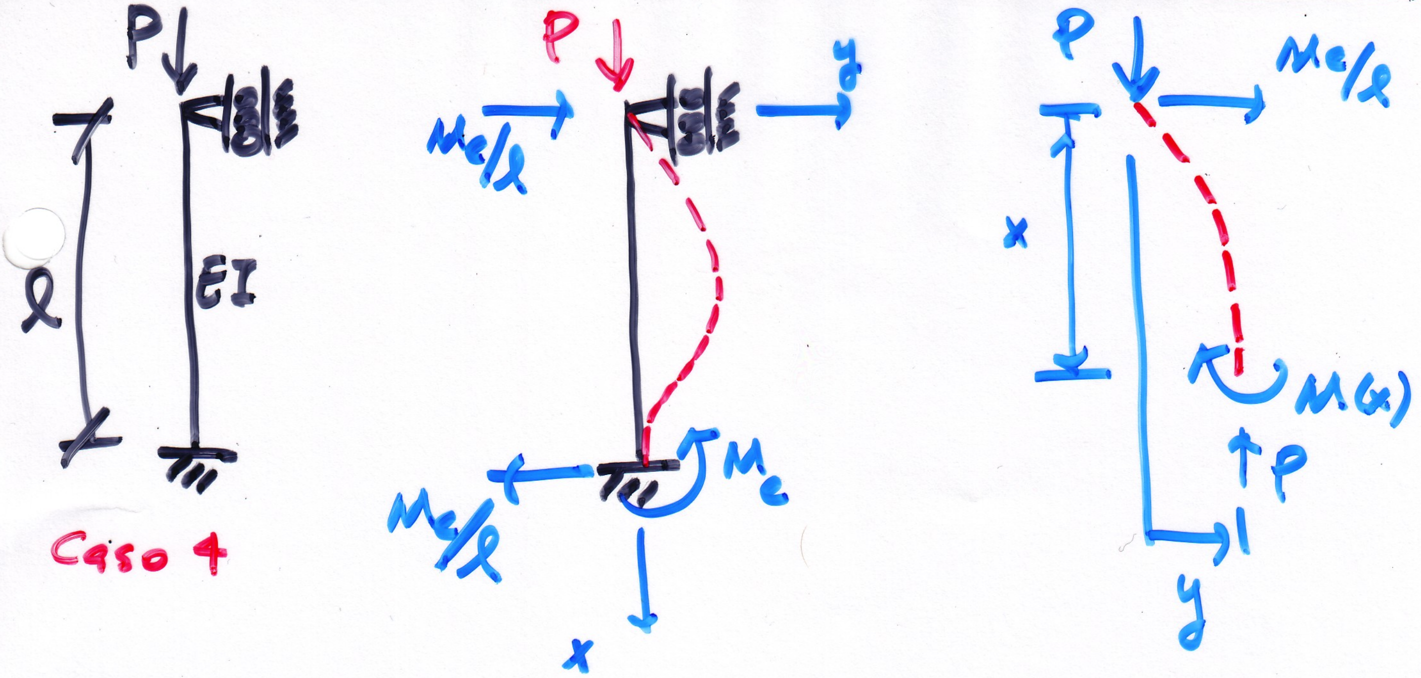
$\Rightarrow \cos K\ell = 1$

$\hookrightarrow K\ell = 2\pi, 4\pi, 6\pi, \ldots, 2n\pi \rightsquigarrow n = 1, 2, 3, \ldots$

$$\boxed{\sqrt{\frac{P}{EI}}}\,\ell = 2n\pi \rightsquigarrow P_{cr} = \frac{(2n\pi)^2 EI}{\ell^2} = \frac{\pi^2 EI}{\left(\frac{1}{2n}\ell\right)^2}$$

$$P_{cr} = \frac{\pi^2 AE}{\left(\frac{1}{2n}\,\ell/r\right)^2} \rightsquigarrow K = \frac{1}{2n} \rightsquigarrow P_{cr} = \frac{\pi^2 AE}{(K\ell/r)^2}$$





$$M(x) + \frac{M_e}{l}x = Py$$

$$EI\frac{d^2y}{dx^2} + Py = \frac{M_e}{l}x$$

$$\frac{d^2y}{dx^2} + K^2y = \frac{M_e}{EIl}x$$

$\hookrightarrow y_h = C_1 \sin Kx + C_2 \cos Kx$

$y_p = C_3 x + C_4 \rightsquigarrow K^2(C_3 x + C_4) = \frac{M_e}{EIl}x \rightsquigarrow C_4 = 0$

$\hookrightarrow C_3 = \frac{M_e}{EIl} \cdot \frac{EI}{P} = \frac{M_e}{Pl}$

$$\Rightarrow y = C_1 \sin Kx + C_2 \cos Kx + \frac{M_e}{Pl}\cdot x$$

Condiciones de Borde:

1) $x = 0$, $y = 0$
2) $x = l$, $dy/dx = 0$
3) $x = l$, $y = 0$

1) $0 = C_2$

2) $0 = K C_1 \cos Kl + \frac{M_e}{Pl} \rightsquigarrow C_1 = -\frac{M_e}{PKl \cos Kl}$

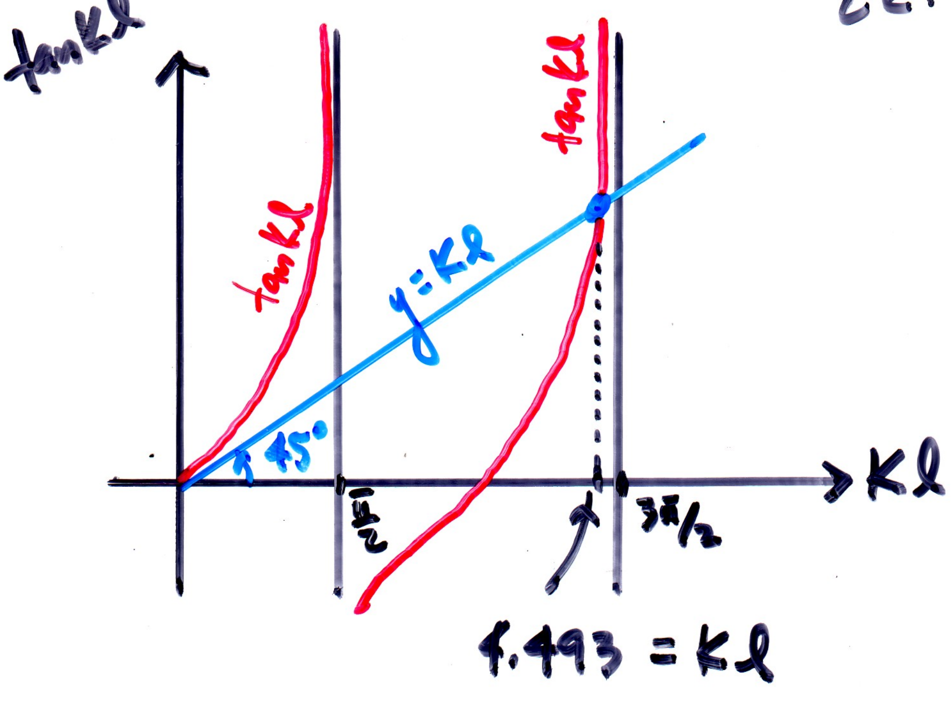
3) $0 = C_1 \sin Kl + \frac{M_e}{P}$

$$\Rightarrow 0 = -\frac{M_e}{PKl} \cdot \frac{\sin Kl}{\cos Kl} + \frac{M_e}{P}$$

$$0 = \frac{M_e}{P}\left(1 - \frac{1}{Kl}\tan Kl\right)$$

$\hookrightarrow 1 - \frac{1}{Kl}\tan Kl = 0 \Rightarrow \boxed{Kl = \tan Kl}$

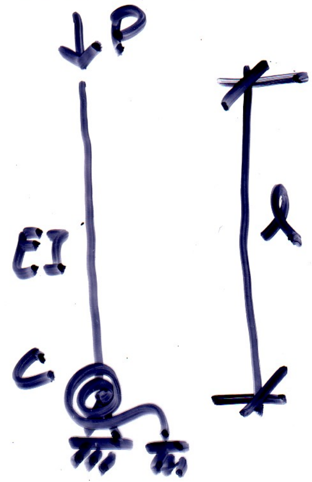
Ec. Trascendental

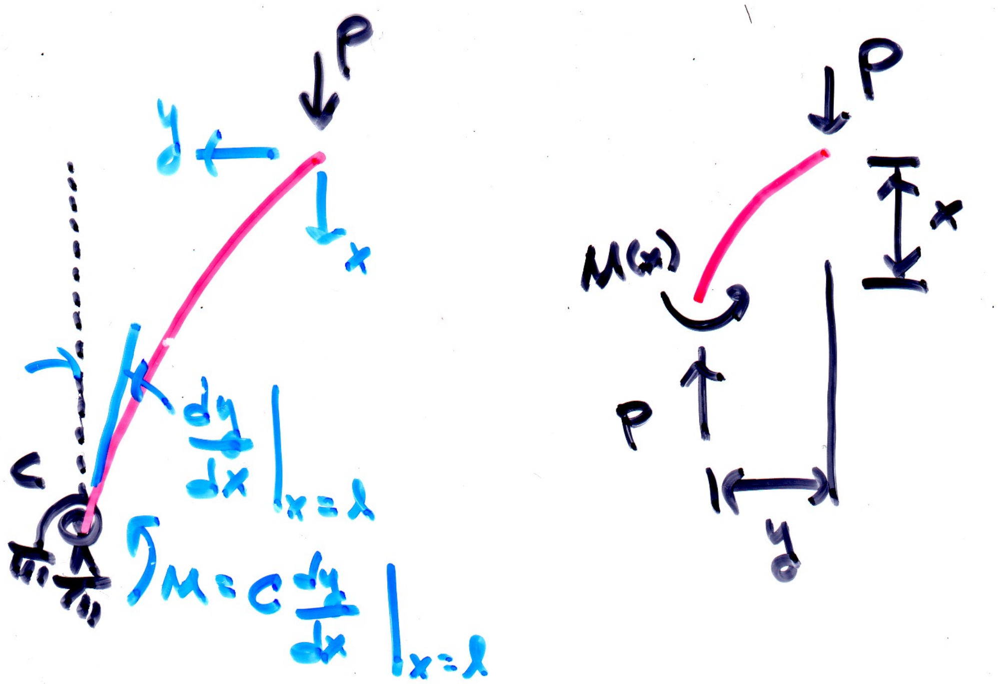


$Kl = 4.493 \rightsquigarrow \sqrt{\frac{P}{EI}}\, l = 4.493$

$$\Rightarrow \boxed{P_{cr} \cong \frac{\pi^2 EI}{(0.7l)^2}} \rightsquigarrow K = 0.7 \rightarrow$$ factor de long. efectiva.

Caso N°5





$$M(x) = Py$$

$$-EI\frac{d^2y}{dx^2} = Py$$

$$\frac{d^2y}{dx^2} + K^2 y = 0 \qquad K^2 = \frac{P}{EI}$$

Solución: $y = C_1 \sin Kx + C_2 \cos Kx$

Condiciones:

① $x = 0$, $y = 0$

② $x = \ell$, $M = Py = C\frac{dy}{dx}\Big|_{x=\ell}$

① $0 = C_2$

$$\Rightarrow y = C_1 \sin Kx$$

$$\frac{dy}{dx} = C_1 K \cos Kx$$

(8)

$$M = Py = C\frac{dy}{dx}$$

$$PC_1 \sin kl = C\,(C_1 K \cos kl)$$

$$C_1\,[\,\overset{EIK^2}{P} \sin kl - CK\cos kl\,] = 0 \;\rightsquigarrow\; C_1 \neq 0$$

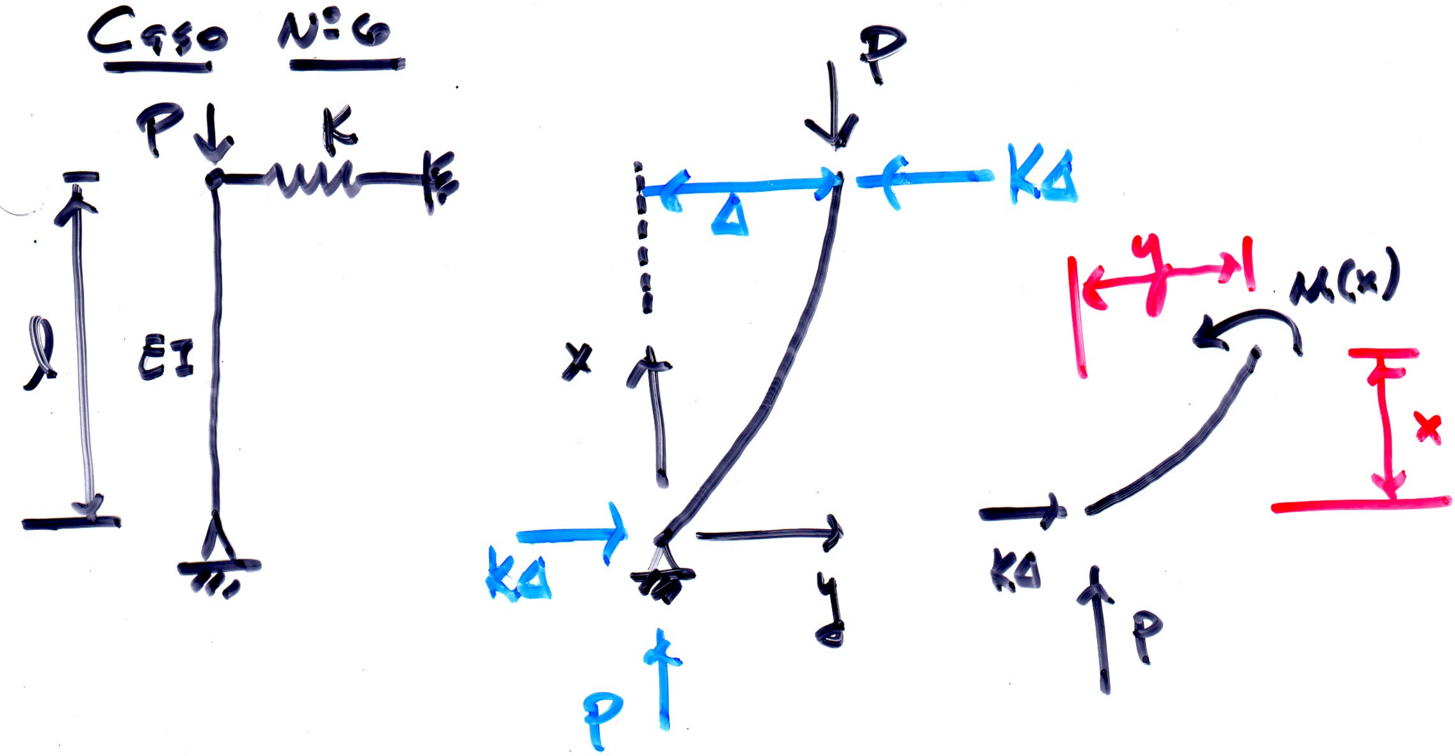
$$\hookrightarrow EIK^2 \sin kl - CK\cos kl = 0$$

$$\frac{EI}{C} K \tan kl - 1 = 0$$

$$\boxed{Kl \tan kl = \frac{Cl}{EI}}$$ Ec. Trascendental

<u>Nota</u>: Podemos esperar que entre más complejos sean los casos, el resultado tendrá q' obtenerse por medio de la solución de una ecuación trascendental.

Caso N°6



$$M(x) + K\Delta X = Py$$

$$-EI\frac{d^2y}{dx^2} + K\Delta X = Py$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{K\Delta X}{EI}$$

$$\frac{d^2y}{dx^2} + \bar{K}^2 y = \frac{K}{EI}\Delta X$$

$$y_h = C_1 \sin \bar{K}x + C_2 \cos \bar{K}x$$

$$y_p = C_3 x + C_4$$

$$\hookrightarrow \bar{K}^2(C_3 x + C_4) = \frac{K\Delta}{EI} x \quad \rightsquigarrow C_4 = 0$$

$$C_3 = \frac{K\Delta}{EI} \cdot \frac{EI}{P} = \frac{K\Delta}{P}$$

Solución: $y = C_1 \sin \underline{k} x + C_2 \cos \underline{k} x + \frac{k\Delta}{P} x$

Condiciones: ① $x = 0$, $y = 0$

② $x = l$, $y = \Delta$

③ $P\Delta = k\Delta l$

$P = kl$

eq. global

① $0 = C_2 \Rightarrow y = C_1 \sin \underline{k} x + \frac{k\Delta}{P} x$

② $\Delta = C_1 \sin \underline{k} l + \frac{k\Delta}{P} l$

③ $\boxed{P_{cr_1} = kl}$ ⟶ La columna gira alrededor de su eje como cuerpo rígido.

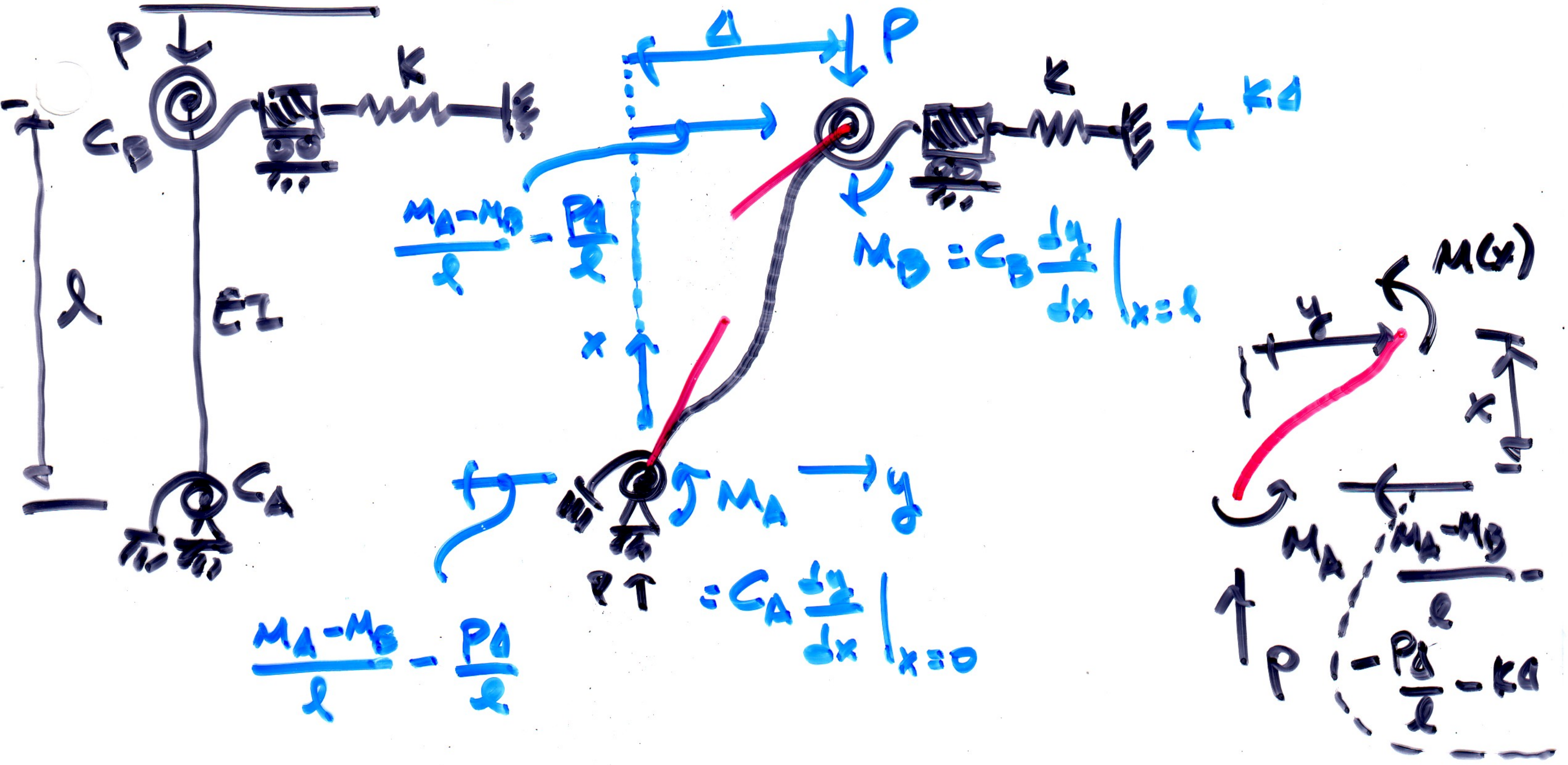
$\Delta = C_1 \sin \underline{k} l + kl \frac{\Delta}{P}$ ← P

$\Delta = C_1 \sin \underline{k} l + \Delta$

↳ $C_1 \sin \underline{k} l = 0 \Rightarrow C_1 \neq 0$

$\sin \underline{k} l = 0 \Rightarrow \underline{k} l = \pi \Rightarrow \boxed{P_{cr_2} = \frac{\pi^2 EI}{l^2}}$

Caso N°7 : El Caso General



$$M(x) + M_A = Py + \left(\frac{M_A - M_B}{l} - \frac{P\Delta}{l}\right)x$$

$$-EI\frac{d^2y}{dx^2} + M_A = Py + \left(\frac{M_A - M_B}{l} - \frac{P\Delta}{l}\right)x$$

$$\frac{d^2y}{dx^2} + K^2 y = \frac{M_A}{EI} - \frac{(M_A - M_B)x}{lEI} + \frac{P\Delta x}{lEI}$$

$$y_h = C_1 \sin Kx + C_2 \cos Kx$$

$$y_p = C_3 x + C_4$$

$$\hookrightarrow K^2(C_3 x + C_4) = \frac{M_A}{EI} - \frac{(M_A - M_B)x}{lEI} + \frac{P\Delta x}{lEI}$$

$$\hookrightarrow \frac{P}{EI} C_4 = \frac{M_A}{EI} \rightarrow C_4 = \frac{M_A}{P}$$

$$\hookrightarrow \frac{P}{EI} C_3 = -\frac{M_A - M_B}{lEI} + \frac{P\Delta}{lEI}$$

$$C_3 = \frac{\Delta}{l} - \frac{M_A}{Pl} + \frac{M_B}{Pl}$$

$$\Rightarrow y = C_1 \sin \sqrt{\tfrac{P}{EI}}\, x + C_2 \cos \sqrt{\tfrac{P}{EI}}\, x + \left(\frac{\Delta}{l} - \frac{M_A}{Pl} + \frac{M_B}{Pl}\right) x + \frac{M_A}{P}$$

Condiciones de Borde:

1) $x = 0, \quad y = 0$

2) $x = l, \quad y = \Delta$

3) $\frac{M_A - M_B}{l} - \frac{P\Delta}{l} = K\Delta \quad \rightsquigarrow$ equilibrio global

4) $M_A = C_A \left.\frac{dy}{dx}\right|_{x=0}$

5) $M_B = C_B \left.\frac{dy}{dx}\right|_{x=l}$

1) $0 = C_2 + \frac{M_A}{P} \quad \rightsquigarrow \quad C_2 = -\frac{M_A}{P}$

2) $\cancel{\Delta} = C_1 \sin \sqrt{\tfrac{P}{EI}}\, l + C_2 \cos \sqrt{\tfrac{P}{EI}}\, l + \cancel{\Delta} + \frac{M_B}{P}$

3) $\frac{M_A}{l} - \frac{M_B}{l} - \frac{P\Delta}{l} = K\Delta$

$$\frac{M_A}{Pl} - \frac{M_B}{Pl} - \frac{\Delta}{l} - \frac{K\Delta}{P} = 0$$

$$\frac{dy}{dx} = \bar{K} C_1 \cos \bar{K} x - \bar{K} C_2 \sin \bar{K} x + \frac{\Delta}{l} - \frac{M_A}{Pl} + \frac{M_B}{Pl}$$

4) $M_A = C_A \left[\bar{K} C_1 + \frac{\Delta}{l} - \frac{M_A}{Pl} + \frac{M_B}{Pl} \right]$

5) $M_B = C_B \left[\bar{K} C_1 \cos \bar{K} l - \bar{K} C_2 \sin \bar{K} l + \frac{\Delta}{l} - \frac{M_A}{Pl} + \frac{M_B}{Pl} \right]$

Simplificando las Ecuaciones:

1) $C_2 + \frac{M_A}{P} = 0 \rightsquigarrow C_2 = -M_A/P$

2) $C_1 \sin \bar{K} l - \frac{M_A}{P} \cos \bar{K} l + \frac{M_B}{P} = 0$

3) $\frac{M_A}{Pl} - \frac{M_B}{Pl} - \Delta\left(\frac{1}{l} + \frac{K}{P}\right) = 0$

4) $C_1 \bar{K} C_A - M_A\left(\frac{C_A}{Pl} + 1\right) + M_B \frac{C_A}{Pl} + \Delta \frac{C_A}{l} = 0$

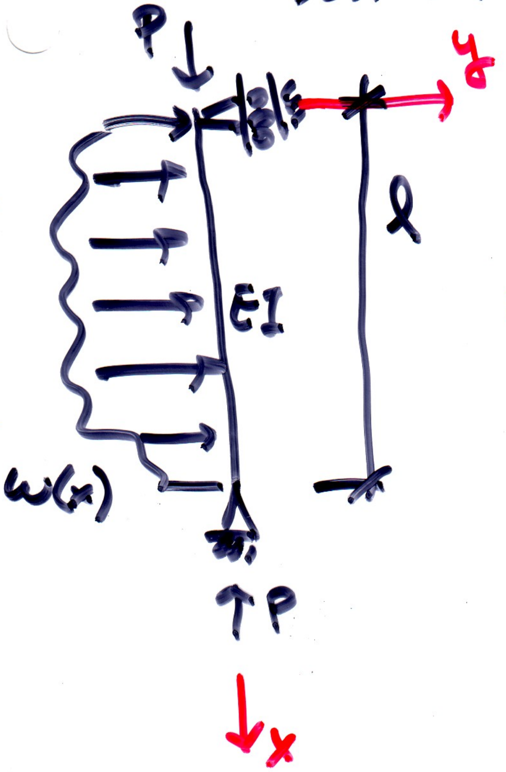
5) $C_1 \bar{K} C_B \cos \bar{K} l + \frac{M_A}{P} \bar{K} C_B \sin \bar{K} l + \Delta \frac{C_B}{l} - M_A \frac{C_B}{Pl} + M_B\left(\frac{C_B}{Pl} - 1\right) = 0$

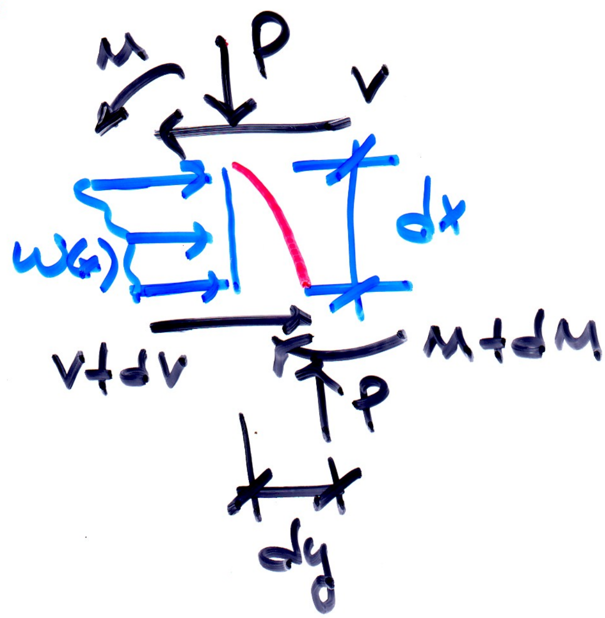
$$
\begin{bmatrix}
\sin K\ell & -\frac{1}{p}\cos K\ell & \frac{1}{p} & 0 \\
0 & \frac{1}{p\ell} & -\frac{1}{p\ell} & -\left(\frac{1}{\ell}+\frac{K}{p}\right) \\
K C_A & -\left(\frac{C_A}{p\ell}+1\right) & C_A/p\ell & C_A/\ell \\
K C_B \cos K\ell & C_B\left(K/p \sin K\ell - \frac{1}{p\ell}\right) & \left(\frac{C_B}{p\ell}-1\right) & C_B/\ell
\end{bmatrix}
\begin{Bmatrix} C_1 \\ M_A \\ M_B \\ \Delta \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
$$

La ecuación trascendental que resuelve este problema resulta al igualar el determinante del sistema de ecuaciones a cero.

Viga - Columna

· Son columnas con cargas transversales.





$$\sum F_y = 0$$

$$w(x)\cdot dx + \cancel{V} + dV - \cancel{V} = 0$$

$$\boxed{w(x) = -\frac{dV}{dx}}$$

$$\sum M = 0$$

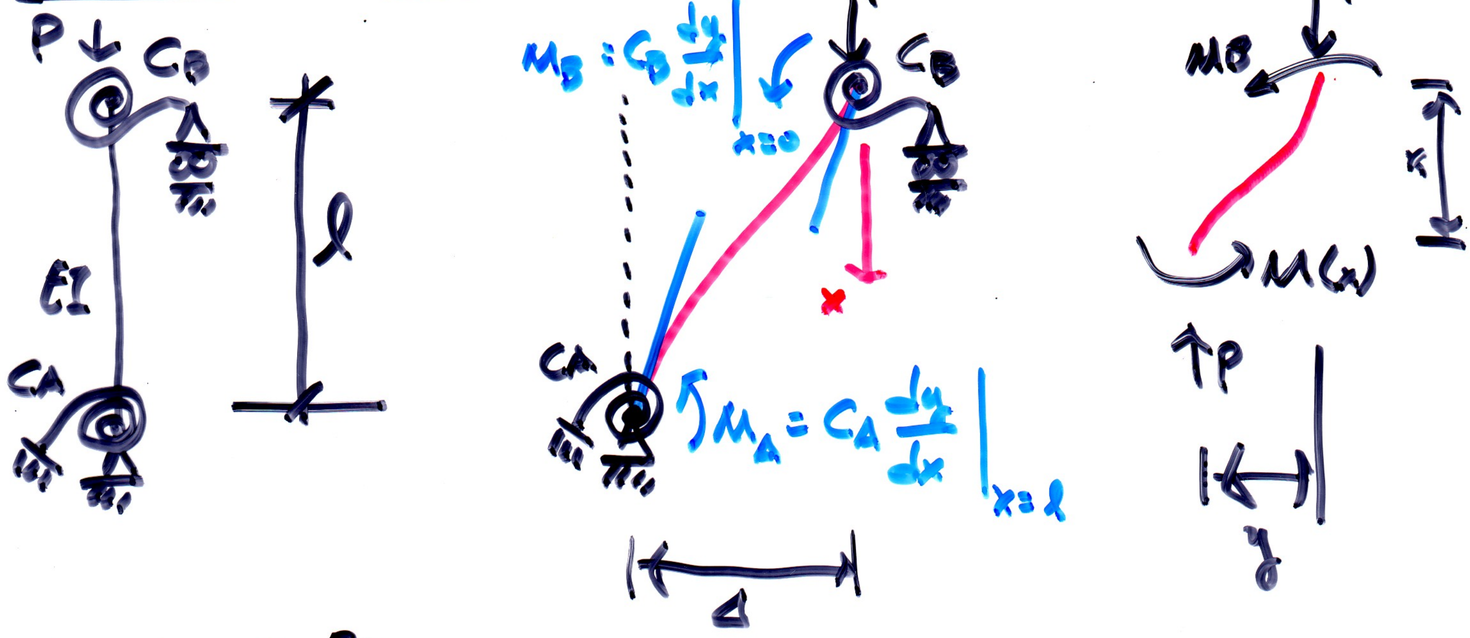
$$\cancel{M} + P\cdot dy + V\cdot dx - \underbrace{w(x)\cdot dx\cdot\frac{dx}{2}}_{\to 0} - \cancel{M} - dM = 0$$

$$\Rightarrow V = \frac{dM}{dx} - P\frac{dy}{dx}$$

$$= \frac{d}{dx}\left(-EI\frac{d^2y}{dx^2}\right) - P\frac{dy}{dx}$$

$$\Rightarrow \boxed{V = -EI\frac{d^3y}{dx^3} - P\frac{dy}{dx}}$$

Soluciones Básicas



$$M(x) + M_B = Py$$

$$-EI \frac{d^2y}{dx^2} + M_B = Py$$

$$\frac{d^2y}{dx^2} + k^2 y = \frac{M_B}{EI}$$

$\hookrightarrow y_h = C_1 \sin kx + C_2 \cos kx$

$\hookrightarrow y_p = C_3 x + C_4$

La Solución General se puede expresar como:

$$y = y_h + y_p$$

$$\Rightarrow y = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4$$

$$y = C_1 \sin kx + C_2 \cos kx + C_3 x + C_4$$

$$\frac{dy}{dx} = C_1 k \cos kx - C_2 k \sin kx + C_3$$

$$\frac{d^2y}{dx^2} = -C_1 k^2 \sin kx - C_2 k^2 \cos kx$$

$$\frac{d^3y}{dx^3} = -C_1 k^3 \cos kx + C_2 k^3 \sin kx$$

Condiciones :

1) $x = 0 \, , \quad y = 0$

2) $x = 0 \, , \quad V = 0$

3) $x = 0 \, , \quad M = M_B$

4) $x = l \, , \quad M = M_A$

1) $0 = C_2 + C_4$

2) $$0 = -EI \frac{d^3y}{dx^3} - P \frac{dy}{dx}$$

$$0 = -EI(-C_1 k^3) - P(C_1 k + C_3) \Rightarrow C_1 \cancelto{0}{(EIk^3 - Pk)} - C_3 P = 0$$

3) $$-EI \frac{d^2y}{dx^2} = C_B \frac{dy}{dx}$$

$$-EI(-C_2 k^2) = C_B (C_1 k + C_3) \Rightarrow C_1 k C_B - C_2 P + C_3 C_B = 0$$

4) $$-EI \frac{d^2y}{dx^2} = C_A \frac{dy}{dx}$$

$$-EI(-C_1 k^2 \sin kl - C_2 k^2 \cos kl) = C_A (C_1 k \cos kl - C_2 k \sin kl + C_3)$$

$$C_1 (P \sin kl - C_A k \cos kl) + C_2 (P \cos kl + C_A k \sin kl) - C_3 C_A = 0$$

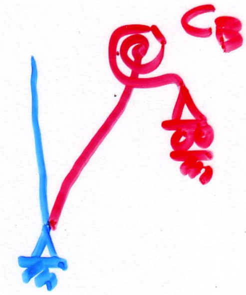
Determinante = 0

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & -P & 0 \\ KC_B & -P & C_B & 0 \\ P\sin Kl - C_A K\cos Kl & P\cos Kl + C_A K \sin Kl & -C_A & 0 \end{vmatrix} = 0$$

$$K C_B P^2 \cos Kl + K^2 C_B C_A P \sin Kl + P^3 \sin Kl - K C_A P^2 \cos Kl = 0$$

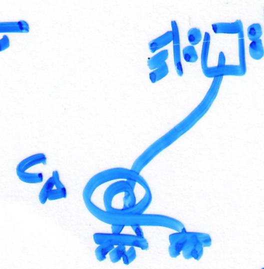
Si $C_B = 0 \Rightarrow \boxed{\dfrac{\tan Kl}{Kl} = \dfrac{C_A}{Pl}}$

Si $C_A = 0 \Rightarrow \boxed{\dfrac{\tan Kl}{Kl} = -\dfrac{C_B}{Pl}}$



Si $C_A = \infty \Rightarrow \boxed{Kl \tan Kl = \dfrac{Pl}{C_B}}$

Si $C_B = \infty \Rightarrow \boxed{Kl \tan Kl = -\dfrac{Pl}{C_A}}$



Si $C_B = 0$, $C_A = \infty \Rightarrow \cos Kl = 0 \Rightarrow P_{cr} = \dfrac{\pi^2 EI}{(2l)^2}$

$\hookrightarrow Kl = \frac{\pi}{2} \longrightarrow$

Si $C_A = C_B = \infty \Rightarrow \sin Kl = 0 \Rightarrow P_{cr} = \dfrac{\pi^2 EI}{l^2}$

$\hookrightarrow Kl = \pi \longrightarrow$